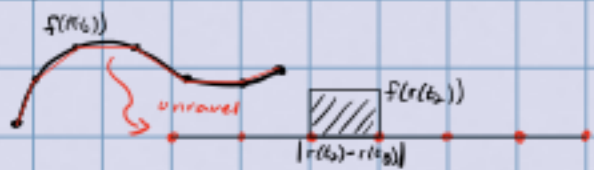
Monday 11/8/21

Section 16.2-16.3 : Line Integrals

IDEA: $f:\mathbb{R}^n\to\mathbb{R}$ a function C a curve in $\mathbb{R}^n$. Understand how f "holds up" along the curve. $[a,b]\xrightarrow{\vec r(t)}\mathbb{R}^n\xrightarrow{f}\mathbb{R}\ldots$, $[a,b]\xrightarrow{f(\vec r(t))}\mathbb{R}$

Picture:



① Approximate the curve piecewise linearly

② "Unravel" approximation to an interval

③ Approximate buildup w/ rectangles having height $f(\text{left endpoint})$ and width $\left|(r(\text{left endpoint})-r(\text{right endpoint})\right|$

④ Limit these approximations by refining the segments

Definition: The line integral of f along curve C parameterized by $\vec r(t)$ on $a\le t\le b$ is:

$$\int_C f\,ds=\int_{t=a}^{b} f(\vec r(t))\,|\vec r\,'(t)|\,dt$$

NB: ① "ds" evokes the idea of arc length…

② if $f=1$, then $s(C)=\int_C 1\,ds=\int_{t=a}^{b}|r'(t)|\,dt=$ arc length of C

Example: Compute $\int_C f\,ds$ for $f(x,y)=x^2+y^2-xy$ and C the upper hemisphere of the unit circle with positive orientation.

Sol: $\int_C f\,ds=\int_{t=a}^{b} f(r(t))\,|\vec r\,'(t)|\,dt$ $\quad\vec r(t)=\langle\cos(t),\sin(t)\rangle$ (because unit circle) $\quad r'(t)=\langle-\sin(t),\cos(t)\rangle$

$0\le t\le\pi$ (upper hemisphere) $\quad|\vec r\,'(t)|=\sqrt{\sin^2(t)+\cos^2(t)}=1$ $\quad f(\cos(t),\sin(t))=f(\vec r(t))=\cos^2(t)+\sin^2(t)-\cos(t)\sin(t)=1-\cos(t)\sin(t)$

$\int_{t=0}^{\pi}(1-\cos(t)\sin(t))\cdot 1\,dt=\begin{pmatrix}u=\cos(t)\\ du=-\sin(t)\,dt\end{pmatrix}\rightarrow\left[t+\frac{1}{2}\cos^2(t)\right]_{t=0}^{\pi}=\left(\pi+\frac{1}{2}(-1)^2\right)-\left(0+\frac{1}{2}(1)^2\right)=\pi$

Definition: For a curve C parameterized by $\vec r(t)$ on $[a,b]$ and $f:\mathbb{R}^n\to\mathbb{R}$ and x_K a variable of f, we define

$$\int_C f\,dx_K=\int_{t=a}^{b} f(\vec r(t))\cdot x_K'(t)\,dt$$

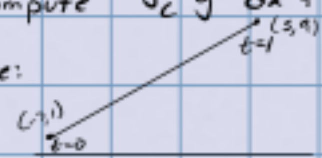
where $x_K'(t)$ is the derivative of the K^{th} term of $\vec r(t)$

$\int_C f\,dx_1=0$ because all change is in the x_2 direction.

Example: Compute $\int_C y^2\,dx+\int_C x\,dy$ for C the line segment oriented from $(-7,1)$ to $(5,9)$

Note: A line segment from A to B we can always parameterize as $\vec r(t)=(1-t)A+tB$ for $0\le t\le 1$

Picture:



Sol: $\vec r(t)=(1-t)\langle-7,1\rangle+t\langle 5,9\rangle=\langle-7+12t,1+8t\rangle$ on $0\le t\le 1$ $\quad\vec r\,'(t)=\langle 12,8\rangle$

$\int_C y^2dx+\int_C x\,dy=\int_{t=0}^{1}(1+8t)^2\cdot 12\,dt+\int_{t=0}^{1}(-7+12t)\cdot 8\,dt=\int_{t=0}^{1}\left(12(1+16t+64t^2)+8(12t-7)\right)dt$

$=4\int_{t=0}^{1}(3+48t+192t^2+24t-14)\,dt=4\int_{t=0}^{1}(-11+72t+192t^2)\,dt=4\left[-11t+36t^2+64t^3\right]_{t=0}^{1}=4(-11+36+64-0)=356$

Definition: The line integral of vector field $\vec v$ along curve C parameterized by $\vec r(t)$ on $[a,b]$ is

$$\int_C\vec v\cdot d\vec r=\int_{t=a}^{b}\vec v(\vec r(t))\cdot r'(t)\,dt\left(=\int_C\vec v\cdot\vec T\,ds\right.$$ where $\vec T(t)$ is unit tangent vector to $\vec r(t)$ i.e. $\left.\vec T(t)=\frac{r'(t)}{|r'(t)|}\right)$

Example: Compute $\int_C\vec v\cdot d\vec r$ for $\vec v=\langle xy,yz,xz\rangle$ and C is the curve parameterized by $\vec r(t)=\langle t,t^2,t^3\rangle$ on $0\le t\le 2$

Sol: $\int_C\vec v\cdot d\vec r=\int_{t=a}^{b}\vec v(\vec r(t))\cdot\vec r\,'(t)\,dt$ $\quad r'(t)=\langle 1,2t,3t^2\rangle$ $\quad\vec v(\vec r(t))=\langle t\,t^2,t^2t^3,t\,t^3\rangle=\langle t^3,t^5,t^4\rangle$

$\int_{t=0}^{2}\langle t^3,t^5,t^4\rangle\cdot\langle 1,2t,3t^2\rangle=\int_{t=0}^{2}(t^3+2t^6+3t^6)\,dt=\left[\frac{1}{4}t^4+\frac{5}{7}t^7\right]_{t=0}^{2}=\frac{16}{4}+\frac{5}{7}\cdot 128-0=\frac{668}{7}$

From Physics: The work done moving a particle along curve C through force field $\vec F$ is $\int_C\vec F\cdot d\vec r$

Example (to do at home): Compute the work done by particle moving along the unit circle counter-clockwise for the quarter-circle through the force field $\vec F=\langle x^2,-xy\rangle$

NB: In example 2: $\int_C y^2dx+\int_C x\,dy$ is often abbreviated $\int_C y^2dx+x\,dy$. In fact, for any curve C we write $\int_C P\,dx+Q\,dy$ to abbreviate $\int_C P\,dx+\int_C Q\,dy$

Question: Noting that these are 1-variable integrals "twisted up" in $\mathbb{R}^n$, Is there an analogue of the FUNDAMENTAL THEOREM OF CALCULUS for use with line integrals?

Bad News: For an integral line $\int_C f\,ds$, there's no reasonable notion of an antiderivative for $f:\mathbb{R}^n\to\mathbb{R}$…

Good News: When $\vec v$ is a conservative vector field, the potential function acts like an antiderivative!

Prop (Fundamental Theorem of Line Integrals [FTLI]): Let $f:\mathbb{R}^n\to\mathbb{R}$ have cts partial derivatives and suppose C is a smooth curve in $\mathbb{R}^n$ parameterized by $\vec r(t)$ on $[a,b]$. Then:

$$\int_C\nabla f\cdot d\vec r=f(\vec r(b))-f(\vec r(a))$$

Proof: We compute $\int_C\nabla f\cdot d\vec r\overset{\text{def of line integral}}{=}\int_{t=a}^{b}\nabla f(\vec r(t))\cdot\vec r\,'(t)\,dt\overset{\text{multivariate chain rule}}{=}\int_{t=a}^{b}\frac{d}{dt}\left[f(\vec r(t))\right]dt\overset{\text{FTC}}{=}\left[f(\vec r(t))\right]_{t=a}^{b}=f(\vec r(b))-f(\vec r(a))$

Example: Compute $\int_C \vec{v}\cdot d\vec{r}$ for $\vec{v} = \langle (1+xy)e^{xy}, x^2e^{xy} \rangle$ on $\vec{r}(t) = \langle \cos(t), 2\sin(t) \rangle$ for $0 \le t \le \pi/2$

Sol: First, verify that $\vec{v}$ is conservative: $\frac{\partial}{\partial y}\left[(1+xy)e^{xy}\right] = (1+xy)e^{xy}x + (1+x)e^{xy} = e^{xy}(2x+x^2y+1)$

$\frac{\partial}{\partial x}\left[(x^2e^{xy})\right] = 2xe^{xy} + x^2(ye^{xy}) = e^{xy}(2x+x^2y)$ Since the two are equivalent $\vec{v}$ is conservative

$f(x,y) = \int \frac{\partial f}{\partial y}\,dy = \int x^2e^{xy}\,dy = xe^{xy} + C(x)$ $\therefore (1+xy)e^{xy} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xe^{xy} + C(x)\right] = e^{xy} + xy + C'(x) = (1+xy)e^{xy} + C'(x)$

$\therefore C'(x) = 0$ yields $C(x) = D_{\text{constant}}$ $\therefore$ We have potential function $f(x,y) = xe^{xy} + D$ choose $D=0$

$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(\pi/2)) - f(\vec{r}(0)) \longleftarrow$ evaluate at home